1. Multiple Choice A One correct answer each.

(a) State the order and the linearity or type of nonlinearity for the following PDE: $(u + u_{xx} + u_{yy})^{19} = u_y + 2022u$.

◯ First order, linear.

◯ First order, nonlinear. It is not quasilinear.

X Second order, nonlinear. It is quasilinear.

◯ Second order, nonlinear. It is not quasilinear.

SOL: Take $(\cdot)^{1/19}$ on both sides.

(b) State the order and the linearity or type of nonlinearity for the following PDE: $(x^2 + 1)\Big(u_{xxx} + \sinh(3y)u_x\Big) = \ln(x^2 + 1)u$.

◯ Second order, nonlinear. It is quasilinear.

◯ Third order, nonlinear. It is quasilinear.

X Third order, linear.

◯ Second order, linear.

SOL: To see this, distribute $(x^2 + 1)$ inside the parenthesis.

(c) The PDE $e^{x+y}u_{xx} + 2e^x u_{xy} - (e^{y-x} - 2)u_{yy} + 7u = \cos(x^2)$, is

◯ nowhere hyperbolic.

◯ parabolic when $x \neq y$.

X parabolic when $x = y$.

◯ everywhere elliptic.

SOL: We compute $\delta = b^2 - ac = e^{2x} + e^{x+y}(e^{y-x} - 2) = e^{2x} + e^{2y} - 2e^{x+y} = (e^x - e^y)^2$.

(d) For which function $f : \mathbb{R} \to \mathbb{R}$ is the following second order linear PDE

$$\frac{1}{4}f(x)u_{xx} + u_{xy} + f(x)u_{yy} - f^2(x)yu_y = u$$

everywhere elliptic?

◯ $f(x) = \sin(x)$.

X $f(x) = \sin(x) + 3$.

- ◯ $f(x) = \frac{1}{2}\sin(x)$.
- ◯ $f(x) = 1$.

SOL: $\delta = b^2 - ac = \frac{1}{4} - \frac{1}{4}f(x)^2$. The PDE is elliptic if $\delta < 0$, that is $f(x)^2 - 1 > 0$. This is the case if $f(x) = \sin(x) + 3$.

2. Multiple Choice B One correct answer each.

(a) By the *transversality condition*, the PDE

$$\begin{cases} (u-y)u_x + (x-y)u_y = \ln(u^2+1), & (x,y) \in \mathbb{R}^2, \\ u(0,y) = y^2, & y \in \mathbb{R}, \end{cases}$$

admits

◯ a global solution.

◯ a local solution for $(x, y) \neq (0, -1)$.

X a local solution for $(x, y) \notin \{(0, 0), (0, 1)\}$.

◯ no solutions at all.

SOL: Here, $a(x, y, u) = u - y$ and $b(x, y, u) = x - y$. The initial curve is parametrized by $\Gamma(s) = (0, s, s^2)$. We can check the transversality condition

$$\det \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} \\ \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix} = \begin{bmatrix} s^2 - s & -s \\ 0 & 1 \end{bmatrix} = s(s-1),$$

which is not zero if $s = y \notin \{0, 1\}$.

(b) Let u be the weak solution of

$$\begin{cases} u_y + (\alpha u - u^2)u_x = 0, & \text{in } (x,y) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = 1, & \text{if } x < 0, \\ u(x,0) = 0, & \text{if } x \geq 0, \end{cases}$$

obtained by imposing the Rankine-Hugoniot condition, where $\alpha \in \mathbb{R}$ is some given constant. Then, u satisfies the entropy condition if

X $\alpha > \frac{4}{3}$.

◯ $\alpha < \frac{4}{3}$.

◯ $\alpha > -\frac{2}{3}$.

◯ $\alpha < \frac{2}{3}$.

SOL: Here $c(u) = \alpha u - u^2$ and $f(u) = \frac{\alpha}{2}u^2 - \frac{u^3}{3}$. We compute $\gamma' = f(1) = \frac{\alpha}{2} - \frac{1}{3}$. The entropy condition is satisfied if

$$0 = c(0) = c(u_+) < \gamma' = \frac{\alpha}{2} - \frac{1}{3} < c(u_-) = c(1) = \alpha - 1.$$

This is the case if $\alpha > \max\{\frac{2}{3}, \frac{4}{3}\} = \frac{4}{3}$.

(c) Denote with $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ the (open) unit disk. For which value of c is the PDE

$$\begin{cases} \Delta u = y^2 + c, & \text{in } D, \\ \partial_\nu u = x, & \text{on } \partial D, \end{cases}$$

solvable?

- ◯ $c = \frac{1}{2}$.
- ◯ $c = 0$.
- X $c = -\frac{1}{4}$.
- ◯ $c = -\frac{\pi}{8}$.

SOL: We have to check the compatibility condition $\int_D \Delta u \, dA = \int_{\partial D} \partial_\nu u \, dl$. Here,

$$\int_D (y^2 + c) \, dA = \int_{\partial D} x \, dl = 0,$$

and therefore, we can compute in polar coordinates

$$\begin{aligned} \pi c &= -\int_D y^2 \, dA = -\int_0^1 r \int_0^{2\pi} (r \sin(\theta))^2 \, d\theta \, dr = -\int_0^1 r^3 \int_0^{2\pi} \sin(\theta)^2 \, d\theta \, dr \\ &= -\frac{1}{4} \int_0^{2\pi} \sin^2(\theta) \, d\theta = -\frac{\pi}{4}. \end{aligned}$$

3. Boxes Fill the white boxes with the final answer. Do not include your computations.

(a) Let $\alpha \in \mathbb{R}$ be a constant. Then, the function

$$u(x,y) = \begin{cases} 1, & \text{if } x \geq \alpha y, \\ 0, & \text{if } x < \alpha y, \end{cases}$$

is a weak solution of the conservation law

$$\begin{cases} u_y + (2u + u^3)u_x = 0, & (x,y) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = 1, & x \geq 0, \\ u(x,0) = 0, & x < 0, \end{cases}$$

if α is equal to

$$\boxed{\alpha = \frac{5}{4}}$$

SOL: We have to check the Rankine-Hugoniot condition for the flux $f(u) = u^2 + \frac{u^4}{4}$: $\alpha = \frac{f(u^+)-f(u^-)}{u^+-u^-} = f(1) = 1 + \frac{1}{4} = \frac{5}{4}$.

(b) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{2x+1}{x^2+y^2+1} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \left\{(x,y) \in \mathbb{R}^2 : 1 < \sqrt{x^2+y^2} < 2\right\}$. The maximum of u in D is equal to

$$\boxed{\max_{\bar{D}} u = \frac{3}{2}}$$

SOL: We apply the maximum principle: $\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial D} \frac{2x+1}{x^2+y^2+1}$. Now, $\partial D = C_1 \cup C_2$ where C_1 is a circle of radius 1, and C_2 is a circle of radius 2. Hence, in polar coordinates

$$\max_{C_1} u = \max_{\theta\in[0,2\pi]} \frac{2\cos(\theta)+1}{\sin(\theta)^2+\cos(\theta)^2+1} = \max_{\theta\in[0,2\pi]} \frac{2\cos(\theta)+1}{2} = \frac{3}{2},$$

and

$$\max_{C_2} u = \max_{\theta\in[0,2\pi]} \frac{4\cos(\theta)+1}{4\sin(\theta)^2+4\cos(\theta)^2+1} = \max_{\theta\in[0,2\pi]} \frac{4\cos(\theta)+1}{5} = \frac{5}{5} = 1.$$

Finally, $\max_{\bar{D}} u = \max_{\partial D} u = \max\{\frac{3}{2}, 1\} = \frac{3}{2}$.

(c) Let $u : [0,1] \times [0,+\infty) \to \mathbb{R}$ be a solution of the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = u(1,t) = e^{-t}\sin(t), & \text{for } t \geq 0, \\ u(x,0) = \sin(\pi x), & \text{for } x \in [0,1]. \end{cases}$$

What is the maximum of u?

$$\boxed{\max u = 1}$$

SOL: Fix $T > 0$, and let $Q_T = [0,T]\times[0,1]$. We apply the maximum principle for the heat equation: $\max_{Q_T} u = \max_{\partial_P Q_T} u$, where $\partial_P Q_T = \{0\}\times[0,1]\cup[0,T]\times\{\{0\}\cup\{1\}\}$. Hence

$$\max_{Q_T} u = \max\left\{ \max_{x\in[0,1]} \sin(\pi x), \max_{t\in[0,T]} e^{-t}\sin(t) \right\} = 1.$$

Since the maximum is independent from T, we have a global maximum in $[0,+\infty)\times[0,1]$.

(d) Consider the one dimensional, homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t)\in\mathbb{R}^2, \\ u(x,0) = g(x), & x\in\mathbb{R}, \\ u_t(x,0) = 1, & x\in\mathbb{R}, \end{cases}$$

where $g(x) = \ln(x^2+1)$ if $|x| < 9$ and $g(x) = 0$ otherwise. Where are the discontinuities of u?

$$\boxed{\{x+2t=9\} \cup \{x-2t=9\} \cup \{x-2t=-9\} \cup \{x+2t=-9\}}$$

SOL: Recall that singularities travel along characteristics. Characteristics are of the form $x \pm 2t = \text{constant}$, and g is singular in $x = \pm 9$.

4. Consider the following PDE in the unit square

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in [0,1] \times [0,1], \\ u_y(x,0) = \alpha, & x \in [0,1], \\ u_y(x,1) = 0, & x \in [0,1], \\ u_x(0,y) = 0, & y \in [0,1], \\ u_x(1,y) = \alpha^2, & y \in [0,1], \end{cases}$$

where $\alpha \in \mathbb{R}$.

(a) Find all values of $\alpha \in \mathbb{R}$ for which the above PDE is solvable.

(b) How many distinct solutions does the above PDE admit? Justify your answer.

(c) Find *all* solutions of the above problem.

If you were not able to solve point (a), set $\alpha = 1$.

(d) Show that the following system

$$\begin{cases} \Delta u(x,y) = 1, & (x,u) \in [0,1] \times [0,1], \\ u_y(x,0) = 1, & x \in [0,1], \\ u_y(x,1) = 0, & x \in [0,1], \\ u_x(0,y) = 0, & y \in [0,1], \\ u_x(1,y) = 1, & y \in [0,1], \end{cases}$$

admits no solution $u \in C^2([0,1] \times [0,1])$.

SOL:

(a) We check the Neumann boundary compatibility condition:

$$0 = \int_{[0,1]^2} \Delta u \, dA = \int_{\partial[0,1]^2} \partial_\nu u \, d\ell = -\int_0^1 \alpha \, d\ell + \int_0^1 \alpha^2 \, d\ell = \alpha(\alpha - 1).$$

The values of α for which the system is solvable are $\alpha = 0$ and $\alpha = 1$.

(b) Recall that if u is a solution of the Laplace equation with Neumann boundary condition, $u + C$ is also a solution for any constant $C \in \mathbb{R}$. Hence, we have an infinite amount of distinct solutions.

(c) When $\alpha = 0$, we have the only possible solutions $u \equiv C \in \mathbb{R}$. We need to solve the system for $\alpha = 1$. Recall that in order to make the slit compatible with the

Neumann boundary compatibility condition, we define $v := u + \beta(x^2 - y^2)$ with $\beta \in \mathbb{R}$ wisely chosen. Splittig $v = v_1 + v_2$ gives

$$\begin{cases} \Delta v_1(x,y) = 0, & (x,y) \in [0,1] \times [0,1], \\ (v_1)_y(x,0) = 1, & x \in [0,1], \\ (v_1)_y(x,1) = -2\beta, & x \in [0,1], \\ (v_1)_x(0,y) = 0, & y \in [0,1], \\ (v_1)_x(1,y) = 0, & y \in [0,1], \end{cases}$$

and

$$\begin{cases} \Delta v_2(x,y) = 0, & (x,y) \in [0,1] \times [0,1], \\ (v_2)_y(x,0) = 0, & x \in [0,1], \\ (v_2)_y(x,1) = 0, & x \in [0,1], \\ (v_2)_x(0,y) = 0, & y \in [0,1], \\ (v_2)_x(1,y) = 1 + 2\beta, & y \in [0,1], \end{cases}$$

Looking at the equation of v_2, we have to set $\beta = -\frac{1}{2}$. Then, $v_2 \equiv C \in \mathbb{R}$, and v_1 solves

$$\begin{cases} \Delta v_1(x,y) = 0, & (x,y) \in [0,1] \times [0,1], \\ (v_1)_y(x,0) = 1, & x \in [0,1], \\ (v_1)_y(x,1) = 1, & x \in [0,1], \\ (v_1)_x(0,y) = 0, & y \in [0,1], \\ (v_1)_x(1,y) = 0, & y \in [0,1], \end{cases}$$

We apply the method of separation of variables: $u(x,y) = X(x)Y(y)$. Imposing the Laplacian equal to zero and looking at the boundary conditions in x we obtain

$$\begin{cases} Y''(y) = cY(y), \\ X''(x) = -cX(x), \quad X'(0) = X'(1) = 0, \end{cases}$$

for some constant $c \in \mathbb{R}$. We have that $c = n^2\pi^2$, $X_n(x) = \cos(n\pi x)$, and

$$Y_n(y) = D_n \cosh(n\pi y) + E_n \cosh(n\pi(y-1)),$$

when $n \geq 1$ and when $n = 0$ we have

$$Y_0(y) = D_0 y + E_0.$$

By the superposition principle

$$v_1(x,y) = D_0 y + E_0 + \sum_{n\geq 1} \cos(n\pi x)\Big(D_n \cosh(n\pi y) + E_n \cosh(n\pi(y-1))\Big).$$

Now,

$$1 = (v_1)_y(x,0) = D_0 + \sum_{n\geq 1} \cos(n\pi x) E_n \sinh(-n\pi) n\pi,$$

and

$$1 = (v_1)_y(x,1) = D_0 + \sum_{n\geq 1} \cos(n\pi x) D_n \sinh(n\pi) n\pi.$$

Hence, $D_0 = 1$ and for $n \geq 1$, $D_n = -E_n = 0$, because the Fourier coefficients of the function $x \mapsto 1$ are $\alpha_n = 0$ if $n \geq 1$ and $\alpha_0 = 1$. Therefore,

$$v_1(x,y) = y + E_0,$$

where E_0 is a general constant. Putting everything together

$$u = v_1 + v_2 - \beta(x^2 - y^2) = y + C + \frac{1}{2}(x^2 - y^2).$$

where $C \in \mathbb{R}$ is an arbitrary constant.

(d) By integration of the Laplacian over the domain we get

$$1 = \int_{[0,1]^2} \Delta u \, dA = \int_{\partial[0,1]^2} \partial_\nu u \, d\ell = 0,$$

which is a contradiction.

5. Consider the PDE

$$\begin{cases} u_y + (u^3+1)u_x = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

where $h(x) = 1$ if $x \leq 0$, $h(x) = (1-x)^{\frac{1}{3}}$ if $0 < x < 1$ and $h(x) = 0$ if $x \geq 1$.

(a) Identify the flux function.

(b) Check the transversality condition: what can we say about the existence of classical solutions? (is it global/local, unique/not unique?).

(c) Compute the critical time of existence y_c.

(d) Solve the above conservation law using the Method of Characteristics.

(e) Find a weak solution everywhere defined.

SOL:

(a) The flux function $f(u)$ is such that $(f(u))_x = (u^2+1)u_x$. Therefore, $f(u) = \frac{u^4}{4} + u$.

(b) We parametrize the initial curve as $\Gamma(s) = \{s, 0, h(s)\}$. We compute

$$\det \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} \\ \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix} = \det \begin{bmatrix} (h(0)^3+1) & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0.$$

Since the transversality condition is satisfied, we have *local* existence and uniqueness of solution.

(c) Our conservation law is of the form $u_y + c(u)u_x = 0$, for $c(u) = (u^3+1)$. The critical time of existence is given by the expression

$$y_c := -\inf\left\{ \frac{1}{(c(h(s)))_s} : s \in \mathbb{R},\ (c(h(s)))_s < 0 \right\}.$$

Now, $c(h(s))$ is equal to 2 if $s \leq 0$, $2-s$ if $0 < s < 1$ and 1 if $s \geq 1$. Therefore, the only interval where the function $(c(h(s)))_s$ does not vanish is when $s \in (0,1)$, obtaining that

$$y_c = -\frac{1}{-1} = 1.$$

(d) Applying the Method of Characteristics we obtain the system of ODEs

$$
\begin{aligned}
x_t(t,s) &= (\tilde{u}(t,s)^3 + 1), & x(0,s) &= s,\\
y_t(t,s) &= 1, & y(0,s) &= 0,\\
\tilde{u}(t,s) &= 0, & \tilde{u}(0,s) &= h(s),
\end{aligned}
$$

that have as general solutions

$$x(t,s) = s + (h(s)^3 + 1)t, \quad y(t,s) = t, \quad \tilde{u}(t,s) = h(s).$$

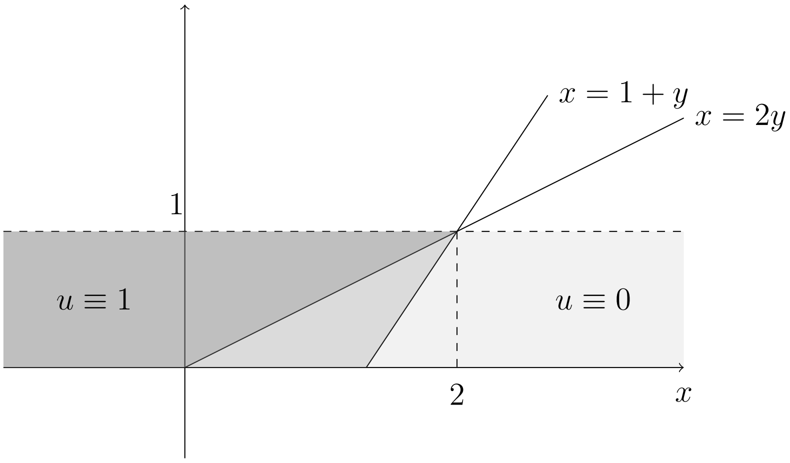
We now have to distinguish between cases:

- If $s \leq 0$, then $x(t,s) = s + 2t$, and hence $s = x - 2y$, implying that $u(x,y) = 1$ when $x - 2y \leq 0$
- If $s \geq 1$, then $x(t,s) = s + t$, and hence $s = x - y$, implying that $u(x,y) = 0$ when $x - y \geq 1$.
- If $0 < s < 1$, then $x(s,t) = s + (1 - s + 1)t$, implying that $s = \frac{x-2y}{1-y}$, giving $u(x,y) = h(\frac{x-2y}{1-y}) = \left(\frac{1-x+y}{1-y}\right)^{\frac{1}{3}}$ when $0 < \frac{x-2y}{1-y} < 1$.

Looking closer to the last inequality $0 < \frac{x-2y}{1-y} < 1$ we see that if $y < 1$, then $2y < x < 1+y$, and if $y > 1$ then $2y > x > 1+y$ (simply by multiplying the inequality by $(1-y)$ on both sides). We obtain the smooth solution for $y < 1$

$$
u(x,y) = \begin{cases} 1, & x \leq 2y, \\ \left(\frac{1-x+y}{1-y}\right)^{\frac{1}{3}}, & 2y < x < 1+y, \\ 0, & x \geq 1+y. \end{cases}
$$

(e) Drawing the functions $x = 2y$ and $x = 1 + y$ we see that the characteristics cross at $(x, y) = (2, 1)$ for the first time.



Expressing the PDE as $u_y + (F(u))_y = 0$, with $F(u) = u + \frac{u^4}{4}$, we can compute the slope of the shock wave taking advantage of the Rankine-Hugoniot condition:

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{-(1 + \frac{1}{4})}{-1} = \frac{5}{4}.$$

Imposing $\gamma(1) = 2$, we obtain that $\gamma(y) = \frac{5}{4}y + \frac{3}{4}$. We can extend the solution of the previous point for $y \geq 1$ setting

$$u(x, y) = \begin{cases} 1, & \text{when } x < \frac{5}{4}y + \frac{3}{4}, \\ 0, & \text{when } x > \frac{5}{4}y + \frac{3}{4}. \end{cases}$$

6. Consider the disk $D := \left\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\right\}$. Let $u = u(x,y)$ be a function twice differentiable in D and continuous in $\bar{D}$, solving

$$\begin{cases} \Delta u(x,y) = 0, & \text{in } D, \\ u(x,y) = g(x,y), & \text{on } \partial D, \end{cases}$$

for some given function g.

(a) Suppose $g(x,y) = x^2 + y^2 + xy$.

i. Compute $u(0,0)$.

ii. Compute $\max_{(x,y)\in\bar{D}} u(x,y)$ and $\min_{(x,y)\in\bar{D}} u(x,y)$.

Hint: recall the trigonometric identity: $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

(b) Suppose now that g is any smooth function such that $g(x,y) \le x^2 - y^2 + y$.

i. Show that $u(\frac{1}{2}, 0) \le \frac{1}{4}$.

ii. Show that if $u(\frac{1}{2}, 0) = \frac{1}{4}$, then $g(x,y) = x^2 - y^2 + y$.

Hint: the function $x^2 - y^2 + y$ is harmonic.

SOL:

(a) For point i. we apply the mean-value principle:

$$\begin{aligned} u(0,0) = \frac{1}{2\pi}\int_{\partial D} g\, d\ell &= \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta)^2 + \sin(\theta)^2 + \sin(\theta)\cos(\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} 1 + \sin(\theta)\cos(\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} 1 + \frac{1}{2}\sin(2\theta)\, d\theta = 1 + \frac{1}{2\pi}\left[-\frac{\cos(2\theta)}{4}\right]_0^{2\pi} = 1. \end{aligned}$$

For point ii. we apply the maximum/minimum principle:

$$\begin{aligned} \max_{\bar{D}} u = \max_{\partial D} u = \max_{\theta\in[0,2\pi]}\left\{\cos(\theta)^2 + \sin(\theta)^2 + \cos(\theta)\sin(\theta)\right\} = \max_{\theta\in[0,2\pi]}\left\{1 + \frac{1}{2}\sin(2\theta)\right\} \\ = 1 + \frac{1}{2} = \frac{3}{2}. \end{aligned}$$

And similarly,

$$\min_{\bar{D}} u = \min_{\partial D} u = \min_{\theta\in[0,2\pi]}\left\{1 + \frac{1}{2}\sin(2\theta)\right\} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(b) For point i. define $w(x,y) := u(x,y) - (x^2 - y^2 + y)$. Then,

$$\begin{cases} \Delta w = 0, & \text{in } D, \\ w \leq 0, & \text{on } \partial D. \end{cases}$$

The maximum principle,

$$\max_{\bar{D}} w = \max_{\partial D} w \leq 0,$$

implies that

$$w(x,y) = u(x,y) - (x^2 - y^2 + y) \leq 0,$$

and hence $u(\frac{1}{2}, 0) \leq \frac{1}{4}$. For point ii. observe that if $u(\frac{1}{2}, 0) = \frac{1}{4}$, then $w(\frac{1}{2}, 0) = 0$, that is w reach its maximum in the interior of D. By the strong maximum principle, $w \equiv 0$, implying that $u = x^2 - y^2 + y$, and hence $g = x^2 - y^2 + y$.